

Optimal Prices and Production Rate in a Closed Loop Supply Chain under Heavy Traffic

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May 26, 2008

Abstract

1 Introduction

Closed loop supply chains that encompass production, distribution, product returns and remanufacture have gained increasing attention recently for both environmental and economic reasons. Remanufacturing typically uses less material and energy than manufacturing, prevents potentially harmful disposal, and retains some of the value added by the original manufacturing process. To the original producer or a third party reprocessor, remanufacturing and reselling products can yield profits both by reducing the cost of providing a functional product and by expanding the market. The status of having been used and refurbished reduces the attractiveness of refurbished products, yet a discounted price can offset this undesirability and create a lower-end market segment of consumers who value the reduced price more than the perceived compromised quality of a refurbished product. A study by Lund (1996) [27] showed that there were 73,000 remanufacturing firms in the U.S. in 1996 with annual industry sales of \$53 billion. Since then, increasing numbers of producers have embraced the opportunities created by closing the supply chain loop.

One of the most important strategic decisions in a closed loop chain is pricing of refurbished products. Their price should be low enough to make them attractive compared with new ones and prevent the inventory of refurbished products from accumulating. On the other hand, too low a price for refurbished products would cannibalize the demand and profits earned by new products. Optimal pricing strategies that coordinate the competition between new and refurbished/remanufactured products (internally or externally) have been analyzed in different contexts. Majumder and Groenevelt (2001) [28] examine the competition between an original equipment manufacturer (OEM) and a local remanufacturer. Debo et. al. (2005) [10] analyze the competition between new and remanufactured products using optimal pricing and production technology selection strategies. They (Debo et. al., 2006 [11]) also extend that study throughout the entire life cycle of the product. Ferrer and Swaminathan (2006) [12] examine both internal and external competition of new and remanufactured products in two periods. Ferguson and Toktay (2006) show that even if remanufacturing itself may not be profitable, manufacturers still may gain strategic benefit by remanufacturing to thwart external remanufacturers' competition.

Collecting or receiving and then reprocessing and reselling products introduces uncertainties in addition to those already present in manufacturing and selling new products. Demand uncertainty may be magnified by the potential for competition between new and remanufactured products. The availability of previously distributed products for remanufacturing is subject to purchasers' decisions on whether and when to return them. Even when the long run average rates of demand, production and returns can be controlled or forecasted accurately, variability along with discreteness of the demand and product flows can create congestion or shortages that reduce the efficiency and economic viability of the closed loop chain.

A large volume of literature has focused on the operational issues in the closed-loop supply chain. Guide and Van Wassenhove (2001) [16] show that a firm could increase its profit by managing the quality of product returns using financial incentives. Guide et al. (2005) [17] report “Inspection-Disposition” policies that helped Hewlett-Packard achieve significant cost savings by testing and identifying returned notebooks that have no defects and refurbishing them in house instead of sending them back to the outsourced design and manufacturing supplier for repair. Savaskan et. al. (2004) [32] propose and compare three models for collecting used products. Other studies include assembly/disassembly operations (Johnson and Wang, 1998 [22]; Ketzenberg et al., 2003 [24]), material planning (Ferrer and Whybark, 2001 [13]), and inventory control (Bayındır et. al., 2003 [4]; Nakashima et. al., 2007 [30]).

Queuing models have been employed in a number of studies to analyze the effectiveness of management policies under steady state conditions (Toktay et al., 2000 [35]; Souza et al., 2002 [33]; Ketzenberg and Souza, 2003 [24]; Guide et al., 2006 [18]). For queueing networks, the conditions under which a steady state exists are well understood: the mean arrival rate to each stage in the system must not exceed the capacity, or maximum processing rate, of that stage. While these conditions ensure that the queue lengths and the waiting times have stable distributions with finite means, they also imply non-negligible idle times in the service facilities. However, many managers prefer to utilize expensive processing resources fully. Because of the stochastic nature of the arrivals and manufacturing/re-manufacturing times, we can only hope to achieve this maximum utilization at the average level. Thus, by setting up a so called “static planning problem” (see also [31]) which is a cost minimization (profit maximization) problem involving deterministic fluid approximations for the queueing network system. The solution to this optimization problem yields restrictions on the parameters of the system, which are commonly known as “heavy traffic conditions”. Since the demand rates are functions of the price of the commodity, this optimization problem turns out to be the optimal price-setting problem. The parameters in our model under the optimal price turns out to be in the heavy traffic parameter regime.

Under the above heavy traffic parameter regime, there is a natural approximation of the queueing network under diffusion scalings. Such approximations are referred as the Brownian approximations (as introduced by Harrison in [20]) or diffusion approximations. Since then several authors (e.g. [5], [9], [29], [8], [2], [31] etc) have used such approximations of physical queueing networks and used techniques from stochastic control theory to obtain good control policies for the queueing networks. In [21], Harrison gives an outline of how such good policies can be obtained (see also [7]), as described below: Obtaining optimal policies involve the following five key steps:

- (a) Formulate a conventional stochastic system model, with an associated dynamic control problem.
- (b) Identify a limiting parameter regime that formalizes the notion of heavy traffic.
- (c) Formulate a Brownian network model, with an associated Brownian control problem that plausibly represents the heavy traffic limit of the original control problem.
- (d) Solve the Brownian control problem and interpret that solution, translating it into a proposed control policy for the original system.
- (e) Show that the proposed policy is asymptotically optimal in the heavy traffic limit, its limiting performance being that associated with the optimal solution of the Brownian control problem.

In this paper, we carry out such an analysis for a re-manufacturing network model, which is two-dimensional. Even if substantial amount of work has been done using different queueing models for one dimensional queueing models (which essentially follows the above 5 steps), there are relatively few works prior to this one (see [9]) which addresses higher dimensional problems (more precisely, the queueing models for which the associated diffusion model is higher dimensional). Also, any such optimization problem requires a cost functional - and two common choices are the long-run average (ergodic) cost and the infinite horizon discounted cost. Here we address optimal control problems for both of these cost functionals. In our model, the system manager is only allowed to control the rate of production of the new items subject to the constraint that the long-run average rate of production satisfying the heavy traffic condition (price-setting equations). These controlled rates differ from this average value by an order of $\frac{1}{\sqrt{n}}$, and can have significant impact of the performance of the network (under diffusion scaling). Such rate controls are sometimes referred to “thin

controls” (see [1], [3], [15]), and in such problems, the associated Brownian control problem becomes a drift control problems for diffusions.

In Section 2 first we formulate the stochastic model and the cost minimization problems (involving two types of costs) for the queueing network model. Next we define the static planning problem using fluid approximations of the processes in the network model. There we obtain a unique solution for the static planning problem which establishes a unique relationship among the average demand rates of new products, refurbished products as well as the manufacturing rates of the new products in terms of the optimal price variable. This relationship enforces the heavy traffic conditions for our queueing network. Our main theorems (Theorem 2.7 and 2.8) describing asymptotically optimal policies are also stated in this section. In Section 3, we address the Brownian control problems (BCP) which are formal approximations of the queueing control problems. Here we obtain the existence of optimal controls for both the BCPs, involving the two types of cost functionals. Section 4 contains weak convergence analysis of the processes in the queueing network to show convergence of these to corresponding processes in the BCPs. We also prove the convergence of cost functionals in queueing problems to those in the BCP as well as provide proofs of the main results. Section 5 contains some numerical analysis in which the proposed optimal controls are computed for specific values of the parameters of the problem. In 6, we discuss **(fill in)**. In the Appendix section at the end, we provide a proof of one real analysis result (Lemma 4.4) that was used in the proofs of asymptotic optimality results.

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In this paper, we consider a simple model of a single firm that can control its production rate of new products but not their price in a competitive market. It produces new products to order. It receives some of these products back after sale and refurbishes them for resale at a price that it chooses to balance the demands for new and refurbished products. The refurbished products are held in inventory. We assume that a customer is willing to wait while a new product is produced to her specifications, but potential buyers of refurbished products are impatient. We use “refurbish” rather than “remanufacture” to denote that products retain their identity and only minimal reprocessing is redone. Unlike the situation where products are fully disassembled and both new and used components may be combined to produce a remanufactured product that is indistinguishable from new, the refurbished products are known to the customer as not new. While stylized, the model captures essential elements of a firm like Dell, Inc., which assembles new products to order, offers a generous return policy, and sells its stock of refurbished products in an online store. The relevant costs are associated with keeping customers waiting for new products, maintaining capacity to manufacture at a given rate, holding refurbished products in inventory, and losing potential refurbished product sales. We derive an asymptotically optimal policy, which consists of the production rate for new products and the relative price of refurbished products, for this closed loop supply chain in heavy traffic. The analysis consists of three main stages. *The next three sentences need work!* First, by considering queue lengths in a fluid scale, we show that it is optimal to set the price and limiting production rate to achieve *approach?* heavy traffic. Second, for demand rates approaching the heavy traffic limit, we identify a production rate dependent on initial conditions that manages the queue lengths in diffusion scale. Finally, we adapt the asymptotic solution to set the prices and production rate for an actual system approaching the heavy traffic limit.

2 Problem Description

We study a simple model of a closed loop supply chain in which a producer manufactures new products to order. Some new products are returned by the customers after evaluation. In this model, we assume that any new product can get returned with probability $\beta \in (0, 1)$. These returned products can no longer be sold as new. Instead, they are inspected, refurbished and placed into inventory to be resold as “refurbished” products. We assume the producer is a price-taker in the market for new products, whose exogenously-determined price is p_N , normalized so that $0 < p_N < 1$. It sets the price for refurbished products, p_R , such that $p_R < p_N$. A consumer who is willing to pay a price of p dollars for a new product is willing to pay at most (δp) for a refurbished product, where $0 < \delta < 1$. Given the prices, the consumer chooses between new and refurbished products to maximize his/her surplus: $\max\{p - p_N, \delta p - p_R, 0\}$. As a result, the normalized demand rates for new and refurbished products, respectively, are (Vorasayan and Ryan, 2006) **(describe this paper a little more here ? also put citation)**:

$$\lambda_N = 1 - \frac{p_N}{1 - \delta} + \frac{p_R}{1 - \delta}, \quad \lambda_R = \frac{p_N}{1 - \delta} - \frac{p_R}{\delta(1 - \delta)}. \quad (2.1)$$

One of the strategic decision variables for the producer is

$$\rho \equiv \frac{p_R}{p_N}. \quad (2.2)$$

Let $\lambda_N(\rho), \lambda_R(\rho)$ denote the rates in (2.1), as functions of ρ (by replacing p_R by (ρp_N)). To guarantee that demands for both products lie in the interval $(0, 1)$, ρ is restricted to the interval $\left(1 - \frac{1 - \delta}{p_N}, \delta\right)$ **(Should we add the relevant calculation somewhere later in the paper?)**.

In our model, the demand for new and refurbished products follow to Poisson processes with the rates $\lambda_N(\rho), \lambda_R(\rho)$ described above (for a chosen value of ρ). We assume that the time required to produce a new product is exponentially distributed with rate $\mu > 0$ and that the manufacturing server is not allowed to idle unless the queue is empty. When a demand for a refurbished item arrives, if such a product is available in inventory then the demand is satisfied; otherwise, the customer is lost. Let $X_1(t)$ denote the length of the new product customer queue and $X_2(t)$ denote the number of refurbished products in inventory at time t . Then, given $X_i(0) = x_i, i = 1, 2$, we model X_1, X_2 as:

$$X_1(t) = x_1 + N_1(\lambda_N(\rho)t) - N_2\left(\int_0^t \mu I_{\{X_1(s) > 0\}} ds\right), \quad (2.3)$$

$$X_2(t) = x_2 + \Phi\left[N_2\left(\int_0^t \mu I_{\{X_1(s) > 0\}} ds\right)\right] - N_3\left(\int_0^t \lambda_R(\rho) I_{\{X_2(s) > 0\}} ds\right), \quad (2.4)$$

where $N_i(\cdot), i = 1, 2, 3$ are independent unit Poisson processes and for any nonnegative integer m , $\Phi(m) = \sum_{k=1}^m \phi_k$, $\{\phi_k\}$ being a sequence of i.i.d Bernoulli(β) random variables. Here $\Phi(m)$ represents the (random) number of products that are returned by customers out of the first m purchased products. See Chapter 6 of [26] to see a more general construction of jump-Markov process, with state space \mathbb{Z} , as a linear combination of time changed versions of unit Poisson process as in (2.3) and (2.4). In our model, we assume that remanufacturing is instantaneous and all returned products are refurbished. Also define processes L_1, L_2 as follows:

$$L_1(t) = \mu \int_0^t I_{\{X_1(s) > 0\}} ds, \quad L_2(t) = \lambda_R(\rho) \int_0^t I_{\{X_2(s) > 0\}} ds. \quad (2.5)$$

Here $L_1(t)$ represents the average number of customers that the server (producing the new items) could not serve because of idleness (no customers of new products in the system) in the interval $[0, t]$. Note that L_1 is defined as the product of average number of new-products that can be manufactured per unit time and

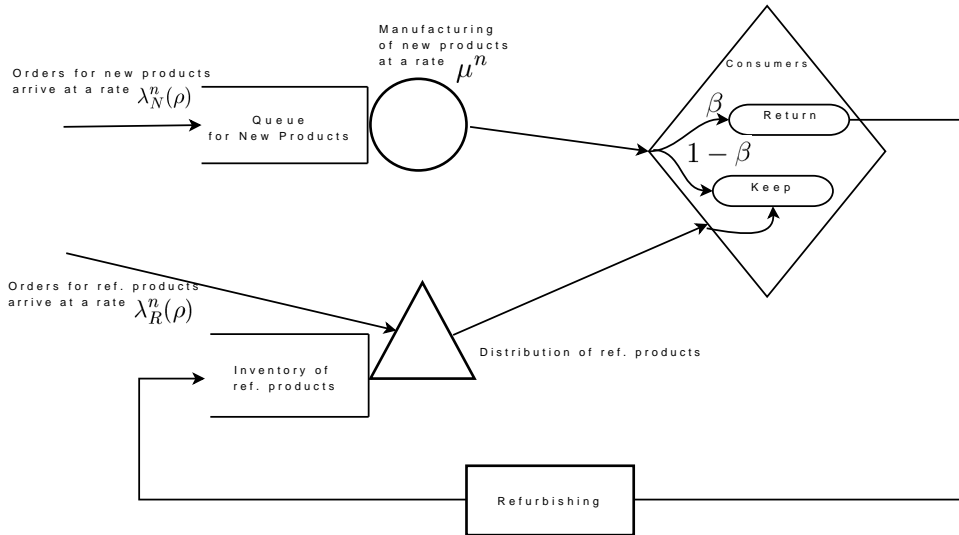


Figure 1: A manufacturing-remanufacturing supply-chain network

the time that the manufacturing server has idled so far. In that sense, it can be thought of as the average number of new products that was not produced because of lack of customers of new products. Using a similar interpretation, $L_2(t)$ captures the average number of lost sales of refurbished products. In our model, we assume that there is no cost for storing refurbished products, and so, ideally, we prefer policies which produce smaller amounts of lost customers of refurbished goods. This fact is used in defining the cost function in (2.9)-(2.10) below (see Remark 2.6 for possible extensions of the model).

As we mentioned earlier, our goal in this paper is to carry out a heavy traffic analysis of the system, and find asymptotically optimal service rate under optimal prices. As it is commonly done for such analysis, we will consider a sequence of networks (indexed by n), each having the same structure, but the parameters of the n -th network depends on the index n , and we will require that as $n \rightarrow \infty$, the system achieves heavy traffic (see Assumption 2.1 and 2.4 below). Hence, from now on, we will consider a sequence of networks indexed by n and all the processes and parameters depend on n (denoted by a superscript n , e.g. $\lambda_N^n(\rho)$, $X_1^n(t)$ etc.). We assume the following basic convergence properties for the parameters of this model:

Assumption 2.1 *For the sequence of parameters described above, we assume the following convergence conditions for any admissible policy. There exists $\theta_i \in \mathbb{R}$, $i = 1, 2, 3$, and $\{\bar{\lambda}_N(\rho) > 0, \rho \in [0, 1]\}$, $\{\bar{\lambda}_R(\rho) > 0, \rho \in [0, 1]\}$ and $\bar{\mu} > 0$ and $x_1, x_2 \geq 0$ such that*

$$\sqrt{n}(\mu^n - \bar{\mu}) \rightarrow \theta_1, \sqrt{n}(\lambda_N^n(\rho) - \bar{\lambda}_N(\rho)) \rightarrow \theta_2, \sqrt{n}(\lambda_R^n(\rho) - \bar{\lambda}_R(\rho)) \rightarrow \theta_3 \text{ for all } \rho \in [0, 1], \quad (2.6)$$

$$\beta\theta_2 = \theta_3 \quad \text{and} \quad \hat{x}_i^n = x_i^n/\sqrt{n} \rightarrow x_i \text{ as } n \rightarrow \infty, i = 1, 2. \quad (2.7)$$

Remark 2.2 *In the assumption above, the first part of (2.7) is a technical assumption which reduces the problem dimension. Because of this, the limiting diffusion control problem is effectively one dimensional, even if the problem is two-dimensional (the state process, scaled (X_1, X_2) , is two-dimensional). This fact helps such analysis of two dimensional system feasible, which is otherwise quite difficult in general. The last part of (2.7) simply says that we do the asymptotic analysis when the initial value of the state system converges. The assumptions in (2.6) states that there are long-run average rates (for arrivals and services) and the parameters of the n -th system converges to those at the \sqrt{n} -rate, θ_i being the convergence rate parameters. All of these are standard assumptions for any heavy traffic analysis.*

As mentioned in the introduction, we will carry out the asymptotic analysis of the diffusion scaled

queueing model. Therefore, we need to define the diffusion scaled before introducing the cost functional. The analysis also involves the “so-called” fluid-scaled processes. For any process $\psi^n(\cdot)$ described here, $\bar{\psi}^n(\cdot)$ and $\hat{\psi}^n(\cdot)$ will denote the fluid- and diffusion-scaled processes respectively, as defined below

$$\bar{\psi}^n(t) = \frac{\psi^n(nt)}{n}, \quad \hat{\psi}^n(t) = \frac{\psi^n(nt)}{\sqrt{n}}, \quad \text{for all } t \geq 0, n = 1, 2, \dots \quad (2.8)$$

In this paper, we analyze two types of cost functionals: the long run average cost (also known as the “ergodic cost”) and the infinite horizon discounted cost, each involving the following components: a control cost for the service rate, a holding cost for the queue for new product (back orders) as well as a linear cost per lost customer of refurbished products. Here we assume that the inventory of the refurbished products does not incur any cost for the manufacturer, hence there is no holding cost for the refurbished products. These components of costs are given in terms of functions $c(\cdot), h(\cdot)$ and a constant penalty rate k , which satisfy the following assumptions.

Assumption 2.3 *Let $h(\cdot)$ and $c(\cdot)$ be nonnegative nondecreasing convex functions defined on $[0, \infty)$ and k be a positive constant. Also assume that there exists $K, N > 0$ such that cost function $c(\cdot)$ satisfies $0 \leq c(x) \leq K(1 + x^N)$ for all $x \geq 0$. In addition, we assume that $h(\cdot)$ is a continuous function.*

The long run average cost (ergodic cost) is given by :

$$\begin{aligned} \hat{I}_0(x_1, x_2, \rho, \{\mu^n\}) &\doteq \liminf_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \frac{1}{T} E \int_0^T \left[\left(h(u^n(\rho)) + c(\hat{X}_1^n(s)) \right) ds + k d\hat{L}_2^n(s) \right] \\ &= \liminf_{n \rightarrow \infty} \limsup_{T \rightarrow \infty} \left[h(u^n(\rho)) + \frac{1}{T} E \int_0^T c(\hat{X}_1^n(s)) ds + k \frac{1}{T} E \hat{L}_2^n(T) \right], \end{aligned} \quad (2.9)$$

and for some fixed discount-factor $\alpha > 0$, the infinite horizon discounted cost is given by:

$$\begin{aligned} \hat{J}_0(x_1, x_2, \rho, \{\mu^n\}) &\doteq \liminf_{n \rightarrow \infty} E \int_0^\infty e^{-\alpha s} \left[\left(h(u^n(\rho)) + \left(\hat{X}_1^n(s) \right)^2 \right) ds + k d\hat{L}_2^n(s) \right] \\ &= \liminf_{n \rightarrow \infty} \frac{h(u^n(\rho))}{\alpha} + E \int_0^\infty e^{-\alpha s} \left[\left(\hat{X}_1^n(s) \right)^2 ds + k d\hat{L}_2^n(s) \right], \end{aligned} \quad (2.10)$$

where $u^n(\rho) = \sqrt{n}(\mu^n - \lambda_R^n(\rho)/\beta), \rho \in [0, 1]$. The first terms in each of (2.9) and ((2.10)) represents the cost of choosing production rate μ relative to the arrival rate for refurbished products (suitably scaled), while the second term is the cost per backorder (of new items) per unit time. The third cost term is the penalty for lost sales of refurbished items. The controls are the price ratio $\rho \in (0, 1)$ as defined in (2.2) and the manufacturing rate sequence $\{\mu^n\}$ in each case (i.e. for each choice of cost functional). Here where x_1, x_2 are the (asymptotic) initial lengths of the backorders of new items and inventory of refurbished items as defined in (2.7). Also note that, we were able to solve the second control problem (infinite horizon discounted cost problem) with backorder cost $c(x) = x^2$.

Even if the cost functions involves only diffusion-scaled processes, to be able to carry out the analysis, one needs to have the fluid system “stable”. Hence we define the static planning problem below, and deduce the conditions for “heavy traffic”. The process of solving the static planning problem also solves the problem of price setting (choosing ρ).

2.1 Static Planning Problem

It is easy to verify from a functional central limit theorem for renewal processes (using Proposition 4.3 and (4.57)-(4.58) and Assumption 2.1) that

$$\bar{X}_1^n \Rightarrow (\bar{\lambda}_N(\rho) - \bar{\mu})e, \quad \bar{X}_2^n \Rightarrow (\beta\bar{\mu} - \bar{\lambda}_R(\rho))e.$$

Also, from the second part of Proposition 4.3, one gets that $\bar{L}_2^n \Rightarrow 0$ as $n \rightarrow \infty$. Static planning problems are formulated by constructing a system where the fluid-scaled processes are replaced by their long-run averages (or fluid limits) and solving a suitable optimization problem involving those averages (see [31]). From the above observations, the natural optimization problem for either choice of the cost function is the following:

$$\min_{\rho, \bar{\mu}} h \left(\bar{\mu} - \frac{\bar{\lambda}_R(\rho)}{\beta} \right) + c (\bar{\lambda}_N(\rho) - \bar{\mu}). \quad (2.11)$$

Clearly, the unique solution to the above static planning problem is given by $(\rho^*, \bar{\mu}^*)$ such that $\bar{\lambda}_N(\rho^*) = \frac{1}{\beta} \bar{\lambda}_R(\rho^*)$ and $\bar{\mu}^* = \bar{\lambda}_N(\rho^*)$. Note that here, from (2.1), there exists such a unique ρ^* . This relation constitutes the heavy traffic condition, and for the rest of the paper, we will assume that all admissible controls satisfy this, in addition to Assumption 2.1.

Assumption 2.4 (Heavy Traffic) *We will assume that any admissible policy satisfies*

$$\bar{\mu}^* = \bar{\lambda}_N(\rho^*) = \frac{1}{\beta} \bar{\lambda}_R(\rho^*).$$

There are different equivalent methods of arriving at the above described heavy traffic assumption needed for analyzing diffusion-scaled systems (see [21], [7]), and it can be verified that those methods also yields the same heavy traffic condition that we have here.

Definition 2.5 (Two Queueing Control Problems) *Under the Assumptions 2.1 and 2.4 above, the price variable is set as ρ^* , which determines the demand-rates of new and refurbished products, as well as the long-run average service rate $\bar{\mu}^*$. A sequence of service rates $\{\mu^n\}$ is said to be admissible if it satisfies the assumptions above with $\rho^*, \bar{\mu}^*$.*

The first queueing control problem is to find an optimal service rate sequence $\{\mu_n^\}$ that minimizes*

$$\hat{I}(x_1, x_2, \{\mu^n\}) \doteq \hat{I}_0(x_1, x_2, \rho^*, \{\mu^n\}) \quad (2.12)$$

over all admissible controls $\{\mu^n\}$.

The second queueing control problem is to find an optimal service rate sequence $\{\mu_n^\}$ that minimizes*

$$\hat{J}(x_1, x_2, \{\mu^n\}) \doteq \hat{J}_0(x_1, x_2, \rho^*, \{\mu^n\}) \quad (2.13)$$

over all admissible controls $\{\mu^n\}$.

Remark 2.6 *In our model, we assume that*

(i) all returned products are selected for remanufacturing.

(ii) Remanufacturing is instantaneous (no delay).

The restriction (i) can be removed in a straightforward manner, if we assume that for every purchased product, there is a probability $\beta_1 \in (0, 1)$ of being returned by the customer, and out of those, each has a probability $\beta_2 \in (0, 1)$ of being selected for remanufacturing. This model is actually covered our model with $\beta = \beta_1 \beta_2$. To extend our model to incorporate delays mentioned in (ii) above, one can replace the first term in defining X_2 in (2.4) by

$$N_4 \left(\int_0^t \gamma \Phi \left(N_2 \left(\int_0^t \mu I_{\{X_1(s) > 0\}} ds \right) \right) \right),$$

where $N_4(\cdot)$ is another unit Poisson process, independent of all the other variables and processes and $\gamma > 0$. In this representation, γ is the “delay rate” for remanufacturing under the assumption that for each returned product selected for remanufacturing, it takes an $\text{Exponential}(\gamma)$ amount of time, before it is ready to be in the inventory. The analysis of such model will be similar to the one considered here, but to keep the model a bit simple, we do not consider such generalizations.

The following are the two main theorems of this article, which shows existence of optimal controls for two queueing control problems described in Definition 2.5 above.

Theorem 2.7 *There exists a $u_1^* \geq 0$ (as in Theorem 3.3) such that for*

$$\mu_1^{n,*} = \frac{1}{\beta} \lambda_R^n(\rho^*) + \frac{u_1^*}{\sqrt{n}}, \quad n = 1, 2, \dots \quad (2.14)$$

is the optimal sequence of service rates for the first queueing control problems defined in Definition 2.5.

Theorem 2.8 *There exists a $u_2^* \geq 0$ (as in Theorem 3.12) such that for*

$$\mu_2^{n,*} = \frac{1}{\beta} \lambda_R^n(\rho^*) + \frac{u_2^*}{\sqrt{n}}, \quad n = 1, 2, \dots \quad (2.15)$$

is the optimal sequence of service rates for the second queueing control problems defined in Definition 2.5.

Note that, in each of the control problems, the above choices in (2.14) and (2.15) are not unique. For example, for the first problem,

$$\tilde{\mu}_1^{n,*} = \lambda_N^n(\rho^*) + \frac{u_1^*}{\sqrt{n}}, \quad n = 1, 2, \dots \quad (2.16)$$

is another such choice. Note that these two choices are asymptotically equivalent, in the sense that the behavior of the diffusion-scaled system under these two choices are same (it is a consequence of the fact that u^n and \tilde{u}^n defined in (4.54) are asymptotically equivalent).

The proofs of the main results (Theorem 2.7 and Theorem 2.8) involve a detailed analysis of the two diffusion control problems that goes with the queueing control problem above. In Section 3, we define the Brownian control problems associated with the two queueing control problems. In this section, we also derive the existence of u^* mentioned in Theorems 2.7 and 2.8 above. The Section 4 contains the proofs of the Theorems 2.7 and 2.8, using the solution of the diffusion control problem and weak convergence techniques. The next section, Section 5 contains numerical examples in which we show how the optimal u_i^* can be obtained for one set of choices of the problem parameters, followed by a discussion in Section 6.

3 Brownian Control Problems

The Brownian control problem (BCP) for a queueing network is formulated by replacing the martingale terms in the scaled queue equations (i.e. \hat{W}_i^n for $i = 1, 2$ above) by a suitable Brownian motions, and constructing a diffusion control problem ([20], [7] etc.). Such control problems often contain useful insights about the queueing control problems, and commonly used for such analysis.

In the next section, we will establish that the sequence $(\hat{X}_1^n, \hat{X}_2^n)$ converge weakly to a two-dimensional process (X_1, X_2) , which is a reflecting diffusion with state space in the first quadrant of \mathbb{R}^2 . Furthermore, (X_1, X_2) satisfies the following stochastic differential equations:

$$\begin{aligned} X_1(t) &= x_1 - ut + \sigma_1 W_1(t) + L_1(t) \\ X_2(t) &= x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1(t) + L_2(t) \end{aligned} \quad (3.17)$$

where $W_1(\cdot)$ and $W_2(\cdot)$ are two standard Brownian motion processes and they are correlated. Their dependence is described by $E[W_1(t)W_2(t)] = -rt$, where $r = \frac{\bar{\mu}\beta}{\sigma_1\sigma_2}$. The constants σ_1 and σ_2 are given by

$\sigma_1 = \sqrt{\lambda_N + \bar{\mu}}$ and $\sigma_2 = \sqrt{\lambda_R + \bar{\mu}\beta(1-\beta)}$. The local-time processes L_1 and L_2 are non-decreasing and satisfies $L_1(0) = L_2(0) = 0$. Furthermore,

$$\int_0^t I_{[X_1(s) > 0]} dL_1(s) = 0$$

and

$$\int_0^t I_{[X_2(s) > 0]} dL_2(s) = 0 \text{ for all } t > 0.$$

The local time processes L_1 and L_2 keep the state processes X_1 and X_2 non-negative, respectively. $X_1(t)$ represents the limiting queue length for the new product at time t and $X_2(t)$ represents the limiting inventory of the refurbished products at time $t > 0$. The constant $u > 0$ in (3.17) is a control parameter which represents the rate of supply of the new product.

The stochastic system described in (3.17) is known as a ‘‘Brownian control system’’. In the following two subsections, we will consider this system described above under the two types of cost structures, and solve the corresponding control problems in each case.

3.1 BCP with long run average cost

First we describe the long-term average cost structure associated with the first control problem. For such a Brownian control system described by (3.17), we consider the long term expected average cost function given by

$$I(x_1, x_2, u) = \limsup_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T (h(u) + c(X_1(s))) ds + k \cdot L_2(T) \right]. \quad (3.18)$$

Here $h(u)$ represents the control cost, $c(X_1(t))$ represents the holding cost for queue length $X_1(t)$ and the constant $k > 0$ represents the penalty per lost customer for refurbished products. Since $h(u)$ is time independent, the cost function in (3.18) can be written as

$$I(x_1, x_2, u) = h(u) + \limsup_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T c(X_1(s)) ds + k \cdot dL_2(t) \right]. \quad (3.19)$$

Throughout, we assume that the cost function $c(\cdot)$ has polynomial growth. In the following discussion, we intend to obtain an optimal control $u^* > 0$ which minimizes $I(x_1, x_2, u)$ over all constant controls $u > 0$.

We can represent the value function of this stochastic control problem by

$$V(x_1, x_2) = \inf_{u > 0} I(x_1, x_2, u).$$

Next, using the equation for $X_1(\cdot)$, we intend to compute $\lim_{T \rightarrow \infty} \frac{1}{T} E \left[\int_0^T c(X_1(s)) ds \right]$ explicitly.

First we introduce the constant $\gamma(u)$ for each control $u > 0$ by

$$\gamma(u) = \frac{2u}{\sigma_1^2} \int_0^\infty c(x) e^{-\frac{2ux}{\sigma_1^2}} dx. \quad (3.20)$$

Since $c(\cdot)$ has polynomial growth, this constant $\gamma(u)$ is finite for each $u > 0$.

Proposition 3.1 *Let (X_1, X_2) satisfy (3.17), then $\lim_{T \rightarrow \infty} \frac{1}{T} E \int_0^T c(X_1(s)) ds$ exists and equal to $\gamma(u)$.*

Proof. Consider the function Q defined on $[0, \infty)$ by

$$Q(x) = \frac{2}{\sigma_1^2} \int_0^x e^{\frac{2u}{\sigma_1^2} r} \int_0^r (\gamma(u) - c(y)) e^{-\frac{2u}{\sigma_1^2} y} dy dr. \quad (3.21)$$

Then Q satisfies

$$\frac{\sigma_1^2}{2}Q'' - uQ' + c(x) = \gamma(u) \text{ for } x > 0 \text{ and } Q'(0) = 0. \quad (3.22)$$

Using (3.20) and (3.21), we also obtain

$$\begin{aligned} Q'(x) &= \frac{2}{\sigma_1^2} e^{\frac{2u}{\sigma_1^2}x} \int_0^x (\gamma(u) - c(y)) e^{-\frac{2u}{\sigma_1^2}y} dy \\ &= \frac{2}{\sigma_1^2} e^{\frac{2u}{\sigma_1^2}x} \int_x^\infty (c(y) - \gamma(u)) e^{-\frac{2u}{\sigma_1^2}y} dy. \end{aligned} \quad (3.23)$$

Therefore, $Q'(x) > 0$ for all x and $Q(0) = 0$, and consequently $Q(x) > 0$ for all $x > 0$.

Next we consider the process $X_1(t)$ in (3.17), and introduce the stopping time τ_N for each $N \geq 1$ as follows:

$$\tau_N = \begin{cases} \inf\{t \geq 0 : X_1(t) \geq N\}, \\ +\infty, \end{cases} \quad \text{if the above set is empty.}$$

We apply Itô's lemma to $Q(X_1(t \wedge \tau_N))$ and use (3.22) to obtain

$$E[Q(X_1(t \wedge \tau_N))] + E \int_0^{t \wedge \tau_N} c(X_1(s)) ds = Q(x) + \gamma(u)E[t \wedge \tau_N]. \quad (3.24)$$

Next we intend to estimate $E[Q_1(t \wedge \tau_N)]$. Using (3.23), we have

$$0 < Q'(x) \leq \left(\frac{2}{\sigma_1^2}\right) e^{\frac{2u}{\sigma_1^2}x} \int_x^\infty c(y) e^{-\frac{2u}{\sigma_1^2}y} dy. \quad (3.25)$$

But $c(\cdot)$ has polynomial growth and therefore for any $0 < \epsilon < \frac{u}{\sigma_1^2}$, we have $0 < c(y) < K_\epsilon e^{\epsilon y}$ for all $y > 0$. Here $K_\epsilon > 0$ is a positive constant which may depend on $\epsilon > 0$. Combining this with (3.25), we obtain $0 < Q'(x) \leq (2/\sigma_1^2) \left(K_\epsilon e^{\epsilon x} / (\frac{2u}{\sigma_1^2} - \epsilon)\right)$, for all $x \geq 0$. By integrating this, we have $0 < Q(x) \leq (2/\sigma_1^2) \left(K_\epsilon e^{\epsilon x} / \epsilon (\frac{2u}{\sigma_1^2} - \epsilon)\right)$ for all $x \geq 0$. Therefore,

$$E[Q(X_1(t \wedge \tau_N))] \leq \tilde{K}_\epsilon E[e^{\epsilon X_1(t \wedge \tau_N)}], \quad (3.26)$$

where \tilde{K}_ϵ is a positive constant. We pick $\delta = 2\epsilon$ and apply Itô's lemma to obtain,

$$\begin{aligned} E[e^{\delta X_1(T \wedge \tau_N)}] &= e^{\delta x_1} + \delta E \int_0^{t \wedge \tau_N} \left(\frac{\sigma_1^2}{2}\delta - u\right) e^{\delta X_1(s)} ds + \delta E[L_1(t \wedge \tau_N)] \\ &\leq e^{\delta x_1} + \delta E[L_1(t \wedge \tau_N)]. \end{aligned}$$

By (3.17), $E[L_1(t \wedge \tau_N)] = uE[t \wedge \tau_N] + E[X_1(t \wedge \tau_N)] - x_1$. Hence, $E[e^{\delta X_1(T \wedge \tau_N)}] \leq e^{\delta x_1} + u \cdot \delta \cdot T + \delta \cdot E[X_1(t \wedge \tau_N)]$. Notice that $E[|X_1(T \wedge \tau_N)|] \leq 1 + E[X_1(T \wedge \tau_N)^2]$ and again with the use of Itô's lemma, we obtain

$$\begin{aligned} E[|X_1(T \wedge \tau_N)|^2] &= x_1^2 + \sigma_1^2 E[t \wedge \tau_N] - 2uE \int_0^{t \wedge \tau_N} X_1(s) ds \\ &\leq x_1^2 + \sigma_1^2 T. \end{aligned}$$

Consequently,

$$E[e^{\delta X_1(T \wedge \tau_N)}] \leq e^{\delta x_1} + \delta x_1^2 + (u\delta + \sigma_1^2 \delta)T + \delta. \quad (3.27)$$

Since $\delta = 2\epsilon$, we have

$$E[e^{\epsilon X_1(T \wedge \tau_N)}] \leq \sqrt{E[e^{\delta X_1(T \wedge \tau_N)}]} \leq \sqrt{e^{\delta x_1} + \delta x_1^2 + \delta(u + \sigma_1^2)T + \delta}$$

Using (3.26), (3.27) and the above estimate, we obtain

$$E[Q(X_1(t \wedge \tau_N))] \leq \tilde{K}_\epsilon \sqrt{e^{2\epsilon x_1} + 2\epsilon x_1^2 + 2\epsilon(u + \sigma_1^2)T + 2\epsilon}$$

We use this estimate together with (3.24) to obtain

$$\begin{aligned} \left| E \int_0^{T \wedge \tau_N} c(X_1(s)) ds - \gamma(u) E[T \wedge \tau_N] \right| &\leq Q(x) + E[Q(X_1(t \wedge \tau_N))] \\ &\leq Q(x) + \tilde{K}_\epsilon \sqrt{e^{2\epsilon x_1} + 2\epsilon x_1^2 + 2\epsilon(u + \sigma_1^2)T + 2\epsilon}. \end{aligned}$$

Next we let τ_N increase to $+\infty$ and divide by T to obtain $\lim_{T \rightarrow \infty} \frac{1}{T} E \int_0^T c(X_1(s)) ds = \gamma(u)$. This completes the proof of the proposition. \blacksquare

In the next lemma, we compare how $E[L_2(T)]$ grows as a function of T .

Lemma 3.2 *Let L_2 be the local time process of X_2 in (3.17), then*

$$\lim_{T \rightarrow \infty} \frac{L_2(T)}{T} = 0 \text{ a.s. and } \lim_{T \rightarrow \infty} \frac{E[L_2(T)]}{T} = 0.$$

Proof. L_1 has the representation (see [19])

$$L_1(t) = \max\{0, -\inf_{0 \leq s \leq t} (x_1 + \sigma_1 W_1(s) - us)\}. \quad (3.28)$$

Consider the Brownian motion $W_1(\cdot)$ and the maximum process M_1 defined by $M_1(t) = \sup_{0 \leq s \leq t} |W_1(s)|$. Then (3.28) implies that $L_1(t) - ut \leq \sigma_1 M_1(t)$, for all $t \geq 0$. Similarly L_2 has the representation $L_2(t) = \max\{0, -\inf_{0 \leq s \leq t} (x_2 + \sigma_2 W_2(s) + \beta us - \beta L_1(s))\}$. Again we introduce $M_2(t) = \sup_{0 \leq s \leq t} |W_2(s)|$. Consequently $0 \leq L_2(t) \leq \sigma_2 M_2(t) + \beta \sigma_1 M_1(t)$, for all $t > 0$. By the properties of the maximum process of Brownian motion (See page 95 and page 112 of [23]), we know that $\lim_{T \rightarrow \infty} \frac{M_1(T)}{T} = \lim_{T \rightarrow \infty} \frac{M_2(T)}{T} = 0$, and $E[M_i(t)] \leq C\sqrt{T}$ for $i = 1, 2$. Therefore, it follows that

$$\lim_{T \rightarrow \infty} \frac{L_2(T)}{T} = 0 \text{ a.s. and } \lim_{T \rightarrow \infty} \frac{E[L_2(T)]}{T} = 0.$$

This completes the proof of the lemma. \blacksquare

Using the previous lemma and Proposition 3.1, we can provide the following explicit representation of the cost function I in (3.19).

$$I(x_1, x_2, u) = h(u) + \gamma(u) = h(u) + \frac{2u}{\sigma_1^2} \int_0^\infty c(x) e^{-\frac{2ux}{\sigma_1^2}} dx \quad (3.29)$$

This expression can be simplified to obtain

$$I(x_1, x_2, u) = \int_0^\infty e^{-y} [h(u) + c\left(\frac{\sigma_1^2}{2u} y\right)] dy. \quad (3.30)$$

The above computations establish the following Theorem.

Theorem 3.3 *The Value function V of the long-run average cost problem described in (3.18)-(3.19) is independent of (x_1, x_2) and it has the following representation*

$$V \equiv V(x_1, x_2) = \inf_{u > 0} \int_0^\infty e^{-y} [h(u) + c\left(\frac{\sigma_1^2}{2u} y\right)] dy.$$

Furthermore, and optimal control u^* is given by $F(u^*) = \min_{u>0} F(u)$ where

$$F(u) = \int_0^\infty e^{-y} \left[h(u) + c\left(\frac{\sigma_1^2}{2u}y\right) \right] dy. \quad (3.31)$$

To compute the optimal u^* we differentiate the above function of u to obtain

$$F'(u) = h'(u) - \frac{\sigma_1^2}{2u^2} \int_0^\infty e^{-y} y c' \left(\frac{\sigma_1^2}{2u} y \right) dy.$$

To find a candidate for u^* , we let $F'(u) = 0$, which yields the following necessary condition:

$$2(u^*)^2 h'(u^*) = \sigma_1^2 \int_0^\infty e^{-y} y c' \left(\frac{\sigma_1^2}{2u^*} y \right) dy.$$

In the case where both $h(\cdot)$ and $c(\cdot)$ are convex twice differentiable increasing functions, the above condition is also sufficient, because

$$F''(u) = h''(u) + \frac{\sigma_1^4}{2u^4} \int_0^\infty e^{-y} y^2 c'' \left(\frac{\sigma_1^2}{2u} y \right) dy + \frac{\sigma_1^2}{u^3} \int_0^\infty e^{-y} y c' \left(\frac{\sigma_1^2}{2u} y \right) dy > 0.$$

For example, when $h(x) = x^m$ and $c(x) = x^n$, where $m \geq 1$ and $n \geq 1$, we obtain that the optimal control u^* is given by

$$u^* = \left(\frac{n}{m} n! \left(\frac{\sigma_1^2}{2} \right)^n \right)^{\frac{1}{m+n}}.$$

3.2 BCP with infinite horizon discounted cost

In the previous section, we have noticed that the expected cost $k \cdot E[L_2(T)]$ which represents the penalty incurred from lost customers for refurbished products during the time interval $[0, T]$ grows at a rate much slower than T as T tends to infinity. In fact, $\lim_{T \rightarrow \infty} \frac{E[L_2(T)]}{T}$ is equal to zero. For this reason, the optimal control policy developed in the previous section is not influenced by this cost component. To capture the effect of the penalty incurred from lost customers for refurbished products, we also consider an infinite horizon discounted cost structure for the same model in (3.17). In this case, cost functional as well as the optimal strategy are affected by the cost component corresponding to the lost customers for refurbished products as well as by the initial data x_1 and x_2 of (3.17).

In our analysis of this cost structure, we use $c(x) = x^2$ to perform explicit computations. A main difficulty in our analysis is to obtain an explicit formula for $E[L_2(T)]$ in this two-dimensional model described in (3.17). For this reason, we are only able to establish a nontrivial optimal control $u^* > 0$ when the initial data (x_1, x_2) belongs to a certain region in \mathbb{R}^2 .

Here we analyze the infinite horizon discounted cost structure given by

$$J(x_1, x_2, u) = E \int_0^\infty e^{-\alpha t} [(h(u) + X_1(t)^2) dt + k \cdot dL_2(t)], \quad (3.32)$$

where $\alpha > 0$ and $k > 0$ are positive constants. We can rewrite this cost functional in the form

$$J(x_1, x_2, u) = \frac{h(u)}{\alpha} + \Phi(x_1, u) + \Psi(x_1, x_2, u), \quad (3.33)$$

where

$$\Phi(x_1, u) = E \int_0^\infty e^{-\alpha t} X_1(t)^2 dt, \quad (3.34)$$

and

$$\Psi(x_1, x_2, u) = k \cdot E \int_0^\infty e^{-\alpha t} dL_2(t). \quad (3.35)$$

The value function for this control problem is given by

$$Q(x_1, x_2) = \inf_{u>0} J(x_1, x_2, u). \quad (3.36)$$

Next, for a given control $u > 0$, we compute $\Phi(x_1, u)$ described in (3.34).

Lemma 3.4 *Let $\Phi(x_1, u)$ be defined by (3.34). Then*

$$\Phi(x_1, u) = \frac{1}{\alpha} \left[\left(x - \frac{u}{\alpha} \right)^2 + \left(\frac{\sigma_1^2}{\alpha} + \frac{u^2}{\alpha^2} \right) \right] - \frac{2u}{\alpha^2 \lambda_1(u)} e^{-\lambda_1(u)x_1}, \quad (3.37)$$

where $\lambda_1(u) = \frac{1}{\sigma_1^2} \left(\sqrt{(u^2 + 2\alpha\sigma_1^2)} - u \right)$.

Proof. Fix $u > 0$, Notice that $\Phi(\cdot, u)$ given in (3.37) satisfies the differential equation

$$\frac{\sigma_1^2}{2} Y'' - uY' - \alpha Y + x^2 = 0 \text{ for } x > 0 \text{ and } Y'(0) = 0. \quad (3.38)$$

Since $u > 0$ is fixed, we relabel $\Phi(x, u)$ by $\Phi(x)$. We apply Itô's lemma to $\Phi(X_1(t))e^{-\alpha t}$ and using (3.38) we obtain

$$E[\Phi(X_1(T \wedge \tau_N))e^{-\alpha(T \wedge \tau_N)}] = \Phi(x_1) - E \int_0^{T \wedge \tau_N} e^{-\alpha t} X_1(t)^2 dt, \quad (3.39)$$

where (τ_N) is a sequence of stopping times defined by

$$\begin{aligned} \tau_N &= \inf\{t \geq 0 : X_1(t) \geq N\} \\ &= +\infty \text{ if the above set is empty} \end{aligned}$$

Using (3.37), we observe that $|\Phi(x)| \leq C(1 + x^2)$, for all $x \geq 0$, where $C > 0$ is a generic constant. Therefore

$$E \left[|\Phi(X_1(T \wedge \tau_N))| e^{-\alpha(T \wedge \tau_N)} \right] \leq C(1 + X_1(T \wedge \tau_N)^2) e^{-\alpha(T \wedge \tau_N)}. \quad (3.40)$$

To prove the assertion of the lemma, we intend to show that $\lim_{T \rightarrow \infty} \lim_{N \rightarrow \infty} E \left[|\Phi(X_1(T \wedge \tau_N))| e^{-\alpha(T \wedge \tau_N)} \right] = 0$ and use it together with (3.39). For this, first we apply Itô's lemma to $X_1(t)^2 e^{-\epsilon t}$ for any fixed $\epsilon > 0$ and obtain the upper bound $E \left[X_1(T \wedge \tau_N)^2 e^{-\epsilon(T \wedge \tau_N)} \right] \leq x_1^2 + \frac{\sigma_1^2}{\epsilon}$ for any $t > 0$. By letting N tend to infinity and using Fatou's lemma, we have $E \left[X_1(t)^2 e^{-\epsilon t} \right] \leq x_1^2 + \frac{\sigma_1^2}{\epsilon}$ for any $t > 0$. Using this estimate together with Itô's lemma for $X_1^4(t) e^{-\epsilon t}$, we obtain

$$\begin{aligned} E \left[X_1^4(T \wedge \tau_N) e^{-\epsilon(T \wedge \tau_N)} \right] &\leq x_1^4 + 6\sigma_1^2 E \int_0^{T \wedge \tau_N} X_1^2(s) e^{-\epsilon s} ds \\ &\leq x_1^4 + 6\sigma_1^2 \int_0^t E[X_1^2(s) e^{-\epsilon s}] ds \\ &\leq x_1^4 + 6\sigma_1^2 \left(x_1^2 + \frac{\sigma_1^2}{\epsilon} \right) t. \end{aligned}$$

Next, we use Hölder's inequality and obtain

$$\begin{aligned} E \left[X_1^2(T \wedge \tau_N) e^{-\alpha(T \wedge \tau_N)} \right] &\leq \left[E \left[X_1^4(T \wedge \tau_N) e^{-\alpha(T \wedge \tau_N)} \right] \right]^{1/2} \left[E \left(e^{-\alpha(T \wedge \tau_N)} \right) \right]^{1/2} \\ &\leq \left[x_1^4 + 6\sigma_1^2 \left(x_1^2 + \frac{\sigma_1^2}{\alpha} \right) T \right]^{1/2} \left[E \left(e^{-\alpha(T \wedge \tau_N)} \right) \right]^{1/2}. \end{aligned}$$

By letting N tend to infinity, $\tau_N \rightarrow \infty$ and thus

$$\lim_{N \rightarrow \infty} E \left[X_1^2(T \wedge \tau_N) e^{-\epsilon(T \wedge \tau_N)} \right] \leq \left[x_1^4 + 6\sigma_1^2 \left(x_1^2 + \frac{\sigma_1^2}{\alpha} \right) T \right]^{1/2} e^{-\frac{\alpha}{2}T}.$$

Combining this with (3.39) and (3.40) we obtain

$$\left| \Phi(x_1) - E \int_0^T e^{-\alpha t} X_1^2(t) dt \right| \leq c \left[e^{-\alpha T} + \left[x_1^4 + 6\sigma_1^2 \left(x_1^2 + \frac{\sigma_1^2}{\alpha} \right) T \right]^{1/2} e^{-\frac{\alpha}{2}T} \right]$$

By letting T tend to infinity, right hand side tends to zero and the assertion of the lemma follows. \blacksquare

For our two-dimensional model described in (3.17), next we intend to consider the functional Ψ given in (3.35). Here we are unable to compute $\Psi(x_1, x_2, u)$ explicitly. Therefore, we intend to obtain an upper bound for the quantity $\Psi(x_1, x_2, u) - \Psi(x_1, x_2, 0)$ in the next lemma. Here $\Psi(x_1, x_2, 0)$ represents the pay-off defined by (3.35) in the case of zero control. To identify the dependence of the processes on the control $u > 0$, we rewrite our model equation (3.17) in the following form:

$$\begin{aligned} X_1^u(t) &= x_1 - ut + \sigma_1 W_1(t) + L_1^u(t) \\ X_2^u(t) &= x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1^u(t) + L_2^u(t), \end{aligned} \quad (3.41)$$

where L_1^u and L_2^u are local time processes of X_1^u and X_2^u respectively. Next we introduce the process \hat{X}^u by

$$\hat{X}^u(t) = x_2 + \beta ut + \sigma_2 W_2(t) + \hat{L}^u(t), \quad (3.42)$$

where $\hat{L}^u(t)$ is the local time process of \hat{X}^u at the origin and hence $\hat{L}^u(t) \geq 0$ for all $t \geq 0$. Notice that (3.42) can be rewritten as

$$\hat{X}^u(t) = x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1^u(t) + \left(\beta L_1^u(t) + \hat{L}^u(t) \right). \quad (3.43)$$

We compare (3.43) with the second equation in (3.41). It is important to recall that the local time process L_2^u is the minimal continuous non-decreasing process which keeps the sum $(x_2 + \beta ut + \sigma_2 W_2(t) - \beta L_1^u(t) + L_2^u(t))$ non-negative. But in (3.43), $\hat{X}^u(t) \geq 0$ for all t and therefore we obtain the inequality

$$L_2^u(t) \leq \beta L_1^u(t) + \hat{L}^u(t), \quad \text{for all } t \geq 0. \quad (3.44)$$

This estimate of $L_2^u(t)$ will be useful in the next proposition.

Proposition 3.5 *Let the initial data (x_1, x_2) be fixed, then the cost functional $J(x_1, x_2, u)$ defined in (3.32) is continuous in the control variable $u > 0$.*

Proof. For $J(x_1, x_2, u)$, we consider the representation (3.33). The function $h(u)$ is continuous in u and by lemma 3.4, $\Phi(x, u)$ is also continuous in u . Therefore, it remains to show that $\Psi(x_1, x_2, u)$ is continuous in the variable u .

For any $u \geq 0$, by (3.35) and Fubini's theorem, we obtain

$$\begin{aligned} \Psi(x_1, x_2, u) &= k \cdot E \left[\int_{t=0}^{\infty} \left(\int_{s=t}^{\infty} \alpha e^{-\alpha s} ds \right) dL_2^u(t) \right] \\ &= k \cdot E \left[\int_{s=0}^{\infty} \alpha e^{-\alpha s} L_2^u(s) ds \right]. \end{aligned}$$

Therefore

$$\Psi(x_1, x_2, u) = \alpha \cdot k \cdot E \int_{t=0}^{\infty} e^{-\alpha t} L_2^u(t) dt = \alpha \cdot k \int_{t=0}^{\infty} e^{-\alpha t} E[L_2^u(t)] dt. \quad (3.45)$$

Next we fix $u \geq 0$ and let $\{u_n\}$ converge to u . We assume that $0 \leq u_n \leq K$ for some fixed constant $K > 0$. It suffices to show that $\lim_{u_n \rightarrow u} \Psi(x_1, x_2, u_n) = \Psi(x_1, x_2, u)$. For each $u \geq 0$, the local time process $L_1^u(t)$ has the representation $L_1^u(t) = \max\{0, \sup_{0 \leq s \leq t} (us - \sigma_1 W_1(s) - x_1)\}$, and therefore, for each $T > 0$ it is evident that $L_1^{u_n}(t)$ converges to $L_1^u(t)$ uniformly on $[0, T]$.

Next $L_2^u(t)$ has the representation $L_2^u(t) = \max\{0, \sup_{0 \leq s \leq t} (\beta L_1^u(s) - \beta us - \sigma_2 W_2(s) - x_2)\}$. Since $u_n \rightarrow u$ and $L_1^{u_n}(t)$ converges uniformly to $L_1^u(t)$ for all $0 \leq t \leq T$, from the above representation it follows that $L_2^{u_n}(t)$ also converges to $L_2^u(t)$ as u_n tends to u a.s. For each u_n , $0 \leq L_1^{u_n}(t) \leq x_1 + Kt + \sigma_1 \sup_{[0,t]} |W_1(s)|$ and $0 \leq L_1^{u_n}(t) \leq x_2 + |\beta|Kt + |\beta|L_1^{u_n}(t) + \sigma_2 \sup_{[0,t]} |W_2(s)|$. Now let $M(t)$ be the process defined by

$$M(t) = |\beta|x_1 + x_2 + 2|\beta|Kt + |\beta|\sigma_1 \sup_{[0,t]} |W_1(s)| + \sigma_2 \sup_{[0,t]} |W_2(s)|.$$

Using Doob's inequality we obtain $E[M(t)^2] \leq C_o(1 + t^2)$, where $C_o > 0$ is a generic constant. Hence $E[M(t)] \leq \sqrt{C_o}(1 + t)$ and $E \int_0^\infty e^{-\alpha t} M(t) dt < \infty$. Since $0 \leq L_2^{u_n}(t) \leq M(t)$ and $L_2^{u_n}(t)$ converges to $L_2^u(t)$ a.s. as u_n tends to u . We can apply Dominated Convergence Theorem to conclude that

$$\lim_{n \rightarrow \infty} E \int_0^\infty e^{-\alpha t} L^{u_n}(t) dt = E \int_0^\infty e^{-\alpha t} L^u(t) dt.$$

Hence $\lim_{n \rightarrow \infty} \Psi(x_1, x_2, u_n) = \Psi(x_1, x_2, u)$ and this completes the proof. ■

Lemma 3.6

Let $\phi(x_1, x_2, u)$ and $\phi(x_1, x_2, 0)$ be as described in (3.35) then

$$\phi(x_1, x_2, u) - \phi(x_1, x_2, 0) \leq \alpha \beta k \int_0^\infty e^{-\alpha t} (E[L_1^u(t)] - E[L_1^0(t)]) dt + k \alpha \int_0^\infty e^{-\alpha t} E[\hat{L}^u(t)] dt + px_2, \quad (3.46)$$

where L_1^u , L_1^0 and \hat{L}^u are the local time processes described in (3.41).

Proof. Using (3.45), we obtain

$$\phi(x_1, x_2, u) - \phi(x_1, x_2, 0) = k\alpha \int_{t=0}^\infty e^{-\alpha t} (E[L_2^u(t)] - E[L_2^0(t)]) dt. \quad (3.47)$$

Next we estimate $(E[L_2^u(t)] - E[L_2^0(t)])$ for each $t \geq 0$. By (3.44), we have $E[L_2^u(t)] \leq \beta E[L_1^u(t)] + E[\hat{L}^u(t)]$. On the other hand, using the second equation of (3.41), we have $E[L_2^0(t)] - \beta E[L_1^0(t)] + x_2 = E[X_2^0(t)] \geq 0$, for all $t \geq 0$. Therefore, $E[L_2^0(t)] \geq \beta E[L_1^0(t)] - x_2$ for all $t \geq 0$. Consequently,

$$(E[L_2^u(t)] - E[L_2^0(t)]) \leq \beta [E[L_1^u(t)] - E[L_1^0(t)]] + E[\hat{L}^u(t)] + x_2 \text{ for all } t \geq 0.$$

Thus, from this estimate in (3.47) we have (3.46). This completes the proof of the lemma. ■

In the next lemma, we compute each of the integrals in (3.46).

Lemma 3.7

(i) For each $u \geq 0$,

$$\alpha \int_0^\infty e^{-\alpha t} E[L_1^u(t)] dt = \frac{1}{2\alpha} (\sqrt{u^2 + 2\alpha\sigma_1^2} + u) e^{-\lambda_1(u)x_1}, \quad (3.48)$$

where $\lambda_1(u) = \frac{1}{\sigma_1^2} (\sqrt{u^2 + 2\alpha\sigma_1^2} - u)$.

(ii) For each $u \geq 0$,

$$\alpha \int_0^\infty e^{-\alpha t} E[\hat{L}^u(t)] dt = \frac{1}{2\alpha} (\sqrt{\beta^2 u^2 + 2\alpha\sigma_2^2} - \beta u) e^{-\lambda_2(u)x_2}, \quad (3.49)$$

where $\lambda_2(u) = \frac{1}{\sigma_2^2} (\sqrt{\beta^2 u^2 + 2\alpha\sigma_2^2} + \beta u)$.

Proof. First notice that $\alpha \int_0^\infty e^{-\alpha t} E[L_1^u(t)] dt = E \int_0^\infty e^{-\alpha t} dL_1^u(t)$. Let $Q(x) = e^{-\lambda_1(u)x}$, where $\lambda_1(u) = \frac{1}{\sigma_1^2} \left(\sqrt{u^2 + 2\alpha\sigma_1^2} - u \right)$. Then Q satisfies

$$\frac{\sigma_1^2}{2} Q''(x) - uQ'(x) - \alpha Q(x) = 0, \quad \text{for } x > 0 \quad \text{and } Q'(0) = -\lambda_1(u).$$

Next, we consider the first equation of (3.41) and apply Itô's lemma to $Q(X_1^u(t))e^{-\alpha t}$ to obtain

$$E[Q(X_1^u(T))]e^{-\alpha T} = Q(x_1) - \lambda_1(u)E \int_0^T e^{-\alpha t} dL_1^u(t).$$

We let T tend to $+\infty$ and obtain

$$Q(x_1) = \lambda_1(u)E \int_0^\infty e^{-\alpha t} dL_1^u(t).$$

Hence (3.48) follows. The proof of (3.49) follows essentially along the same steps by using $Q(x) = e^{-\lambda_2(u)x}$ where $\lambda_2(u)$ is given in (3.49) and the process \tilde{X} in (3.42). This completes the proof. ■

The following proposition follows from the lemmas 3.6 and 3.7.

Proposition 3.8

$$\begin{aligned} \phi(x_1, x_2, u) - \phi(x_1, x_2, 0) &\leq \frac{k\beta}{2\alpha} \left[\left(\sqrt{u^2 + 2\alpha\sigma_1^2} + u \right) e^{-\lambda_1(u)x_1} - \sqrt{2\alpha\sigma_1^2} e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1} \right] \\ &+ \frac{k}{2\alpha} \left[\left(\sqrt{\beta^2 u^2 + 2\alpha\sigma_2^2} - \beta u \right) e^{-\lambda_2(u)x_2} \right] + px_2. \end{aligned} \quad (3.50)$$

The proof of this proposition is straightforward and therefore we omit it.

Remark 3.9 *The estimates we have obtained in the proof of the above proposition and its proof also yields the following upper bound of the cost functional $J(x_1, x_2, u)$ defined in (3.33):*

$$J(x_1, x_2, u) < \frac{h(u)}{\alpha} + \Phi(x_1, u) + k \left[\frac{\beta}{\lambda_1(u)} e^{-\lambda_1(u)x_1} + \frac{1}{\lambda_2(u)} e^{-\lambda_2(u)x_2} \right]. \quad (3.51)$$

In the next proposition, we obtain a sufficient condition which guarantees $J(x_1, x_2, u) < J(x_1, x_2, 0)$ where the cost functional J is defined in (3.33).

Proposition 3.10 *Let (x_1, x_2) be the initial data in (3.17). If there is a control $u \geq 0$ which satisfies*

$$\left[\alpha \left(k\beta - \frac{2u}{\alpha^2} \right) \frac{1}{\lambda_1(u)} e^{-\lambda_1(u)x_1} + \alpha k \frac{1}{\lambda_2(u)} e^{-\lambda_2(u)x_2} + \alpha k x_2 \right] < \left[\frac{\alpha k \beta \sigma_1}{\sqrt{2\alpha}} e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1} + \frac{2u}{\alpha} \left(x_1 - \frac{u}{\alpha} \right) - h(u) \right],$$

where $\lambda_2(u)$ and $\lambda_1(u)$ are described in (3.49) and (3.37) respectively, then $J(x_1, x_2, u) < J(x_1, x_2, 0)$.

Proof. Using (3.33) we observe that $J(x_1, x_2, u) < J(x_1, x_2, 0)$ if and only if

$$\phi(x_1, x_2, u) - \phi(x_1, x_2, 0) < [\Phi(x_1, 0) - \Phi(x_1, u)] - \frac{h(u)}{\alpha}. \quad (3.52)$$

Next we can use the estimate (3.50) in Proposition 3.8. Therefore, the inequality

$$\left[k\beta \left(\frac{1}{\lambda_1(u)} e^{-\lambda_1(u)x_1} - \frac{\sqrt{2\alpha\sigma_1^2}}{2\alpha} e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1} \right) + \frac{k}{\lambda_2(u)} e^{-\lambda_2(u)x_2} + px_2 \right] < [\Phi(x_1, 0) - \Phi(x_1, u)] - \frac{h(u)}{\alpha}$$

implies the inequality in (3.52). Using (3.37) and following a straightforward computation, we obtain

$$\left(k\beta - \frac{2u}{\alpha^2} \right) \frac{1}{\lambda_1(u)} e^{-\lambda_1(u)x_1} + \frac{k}{\lambda_2(u)} e^{-\lambda_2(u)x_2} + px_2 < \frac{2u}{\alpha^2} \left(x_1 - \frac{u}{\alpha} \right) + \frac{k\beta\sigma_1}{\sqrt{2\alpha}} e^{-\frac{\sqrt{2\alpha}}{\sigma_1}x_1} - \frac{h(u)}{\alpha}.$$

This inequality is same as (3.51) and hence the result follows. \blacksquare

Remark 3.11

1. If x_2 and u remain fixed and x_1 tends to infinity then the right hand side of the inequality in (3.51) tends to infinity while the left hand side of (3.51) tends to $\frac{\alpha k}{\lambda_2(u)} e^{-\lambda_2(u)x_2} + \alpha k x_2$. Therefore, for fixed x_2 and u , large values of x_1 satisfy (3.51).

2. If $2u_0 > \alpha^2 k\beta$ and if there is a point (\bar{x}_1, x_2) that satisfies

$$\frac{2u_0}{\alpha} \left(\bar{x}_1 - \frac{u_0}{\alpha} \right) - h(u_0) - \alpha k x_2 > \frac{1}{k} \left(\sqrt{\beta^2 u_0^2 + 2\alpha\sigma_2^2} - \beta u_0 \right),$$

then the above inequality holds for all $x_1 \geq \bar{x}_1$. It is a straightforward matter to check that the assumption of Proposition 3.10 is true. Hence $J(x_1, x_2, u) < J(x_1, x_2, 0)$ for all $x_1 \geq \bar{x}_1$.

Next, we introduce the region

$$\mathcal{A} = \{(x_1, x_2) : \text{There exists } u > 0 \text{ where } (x_1, x_2, u) \text{ satisfies (3.51)}\}. \quad (3.53)$$

This set \mathcal{A} is non-empty as explained in the above remarks.

Theorem 3.12 *Let the initial data (x_1, x_2) of (3.17) belongs to the set \mathcal{A} in (3.53). Then there is an optimal control $u^* > 0$ such that $Q(x_1, x_2) = J(x_1, x_2, u^*)$, where Q is the value function defined in (3.36).*

Proof. We obtain $J(x_1, x_2, u) < J(x_1, x_2, 0)$ for some $u > 0$ by using Proposition 4.6. On the other hand by (3.33), $J(x_1, x_2, u) > \frac{h(u)}{\alpha}$ for all $u > 0$. Since $h(\cdot)$ is strictly increasing and $\lim_{u \rightarrow +\infty} h(u) = +\infty$, there exists $u_2 > 0$ such that $\frac{h(u)}{\alpha} > J(x_1, x_2, 0)$ for all $u > u_2$. Consequently $\inf_{u>0} J(x_1, x_2, u) = \inf_{0 < u < u_2} J(x_1, x_2, u)$. By Proposition 3.5, $J(x_1, x_2, \cdot)$ is continuous in u variable. Therefore, there exists a u^* such that $0 < u^* < u_2$ and $J(x_1, x_2, u^*) = \inf_{u>0} J(x_1, x_2, u)$. This completes the proof. \blacksquare

4 Asymptotic optimality

In this section, we will prove the main theorems of the paper, Theorem 2.7 and 2.8, which prove that our proposed policies for the queueing control problem (for each of the control problems) are asymptotically optimal. Throughout this section, $\rho^*, \bar{\mu}^*$ are fixed and for simplicity of notation, we will denote $\bar{\mu}^*$ by $\bar{\mu}$ and omit ρ^* from all notations, i.e. denote $\lambda_N^n(\rho^*), \bar{\lambda}_N(\rho^*), \lambda_R^n(\rho^*), \bar{\lambda}_R(\rho^*)$ by $\lambda_N^n, \bar{\lambda}_N, \lambda_R^n, \bar{\lambda}_R$ respectively. Also, throughout this section $e \in \mathcal{D}([0, \infty), \mathbb{R})$ will denote the identity function, i.e. $e(t) = t$ for all $t \geq 0$.

4.1 Scaled processes and a Skorohod map

We have defined the two types of scalings for the various processes considered in this paper in (2.8) above. Here we obtain explicit forms of some of the processes that are relevant for our analysis. Recall the definition of u^n in (2.9)-(2.10) above. We define another similar quantity \tilde{u}^n below. Note that by Assumption 2.1 these two quantities are asymptotically equivalent: There exists $u \geq 0$ such that

$$u^n = \sqrt{n} \left(\mu^n - \frac{\lambda_R^n}{\beta} \right) \rightarrow u, \quad \tilde{u}^n = \sqrt{n} (\mu^n - \lambda_N^n) \rightarrow u, \quad \text{as } n \rightarrow \infty. \quad (4.54)$$

Next we define a few other processes that will be useful in our analysis. Let

$$\begin{aligned} \hat{M}_1^n(t) &= \frac{1}{\sqrt{n}} [N_1^n(n\lambda_N^n t) - n\lambda_N^n t], \quad \hat{M}_2^n(t) = \frac{1}{\sqrt{n}} \left[N_2^n \left(n \int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right) - n \int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right], \\ \hat{M}_3^n(t) &= \frac{1}{\sqrt{n}} \left[\Phi^n \left(N_2^n \left(n \int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right) \right) - n\beta \int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right], \\ \hat{M}_4^n(t) &= \frac{1}{\sqrt{n}} \left[N_3^n \left(n \int_0^t \lambda_R^n I_{\{\hat{X}_2^n(s) > 0\}} ds \right) - n \int_0^t \lambda_R^n I_{\{\hat{X}_2^n(s) > 0\}} ds \right], \end{aligned} \quad (4.55)$$

$$\hat{W}_1^n(t) = \hat{M}_1^n(t) - \hat{M}_2^n(t), \quad \hat{W}_2^n(t) = \hat{M}_3^n(t) - \hat{M}_4^n(t), \quad (4.56)$$

for all $t \geq 0, n = 1, 2, \dots$. Hence, from the definition of the processes in (2.3)-(2.4), (2.5), and the definition of diffusion scaled processes in (2.8), we have using a simple change of variable formula (in particular, $\int_0^{nt} g(s) ds = n \int_0^t g(ns) ds$) that for all $n = 1, 2, \dots, t \geq 0$,

$$\begin{aligned} \hat{X}_1^n(t) &= \hat{x}_1^n + \frac{1}{\sqrt{n}} N_1^n(n\lambda_N^n t) - \frac{1}{\sqrt{n}} N_2^n \left(n \int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right) \\ &= \hat{x}_1^n - \tilde{u}^n t + \hat{W}_1^n(t) + \hat{L}_1^n(t), \end{aligned} \quad (4.57)$$

$$\begin{aligned} \hat{X}_2^n(t) &= \hat{x}_2^n + \frac{1}{\sqrt{n}} \Phi^n \left(N_2^n \left(n \int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right) \right) - \frac{1}{\sqrt{n}} N_3^n \left(n \int_0^t \lambda_R^n I_{\{\hat{X}_2^n(s) > 0\}} ds \right) \\ &= \hat{x}_2^n + \beta u^n t + \hat{W}_2^n(t) - \beta \hat{L}_1^n(t) + \hat{L}_2^n(t), \end{aligned} \quad (4.58)$$

$$\text{where} \quad \hat{L}_1^n(t) = \mu^n \sqrt{n} \int_0^t I_{\{\hat{X}_1^n(s) = 0\}} ds, \quad \hat{L}_2^n(t) = \lambda_R^n \sqrt{n} \int_0^t I_{\{\hat{X}_2^n(s) = 0\}} ds. \quad (4.59)$$

The proof of asymptotic optimality uses the following maps and their properties.

Lemma 4.1 (A two-dimensional Skorohod Map) *Let $u_1, u_2, \beta \geq 0$ and let $u = (u_1, u_2)$. For each $x_1, x_2 \geq 0$ and $w = (w_1, w_2) \in \mathcal{D}([0, \infty), \mathbb{R}) \times \mathcal{D}([0, \infty), \mathbb{R})$ with $w_i(0) \geq 0, i = 1, 2$, there exists unique $q_i, \ell_i \in \mathcal{D}([0, \infty), \mathbb{R}), i = 1, 2$, satisfying the following properties.*

- (i) $q_1(t) = x_1 - u_1 t + w_1(t) + \ell_1(t) \geq 0, \quad \forall t \geq 0,$
- (ii) $q_2(t) = x_2 + \beta u_2 t + w_2(t) - \beta \ell_1(t) + \ell_2(t) \geq 0, \quad \forall t \geq 0,$
- (iii) $\ell_i(\cdot)$ is nondecreasing, $\ell_i(0) = 0$ and $\int_0^\infty q_i(t) d\ell_i(t) = 0$ for $i = 1, 2$.

Define the following maps $\Gamma_i^u, \hat{\Gamma}_i^u, i = 1, 2$ as follows: for a given w as above, let $\Gamma_i^u(w) = q_i, \hat{\Gamma}_i^u(w) = \ell_i, i = 1, 2$. We will denote call the map $(\Gamma_i^u(\cdot), \hat{\Gamma}_i^u(\cdot)) : i = 1, 2$ as the Skorohod map relevant for this problem.

Proof of the existence of the above map is straightforward. For $x \in \mathcal{D}([0, \infty), \mathbb{R})$ with $x(0) \geq 0$, define the following maps

$$\phi(x)(t) = x(t) + \psi(x)(t), \quad \text{where } \psi(x)(t) = - \inf_{0 \leq s \leq t} \min\{x(s), 0\}, \quad \text{for } t \geq 0. \quad (4.60)$$

The above maps are called one-dimensional Skorohod maps in $[0, \infty)$ (see [34], [25]). Using the above maps it is easy to verify that if $w_i \in \mathcal{D}([0, \infty), \mathbb{R})$, $i = 1, 2$ the following representations hold for the Skorohod maps defined in Lemma 4.1.

$$\begin{aligned} \Gamma_1^u(w) &= \phi(x_1 - ue + w_1), \quad \hat{\Gamma}_1^u(w) = \psi(x_1 - ue + w_1), \\ \Gamma_2^u(w) &= \phi\left(x_2 + \beta ue + w_2 - \beta \hat{\Gamma}_1^u(w)\right), \quad \hat{\Gamma}_2^u(w) = \psi\left(x_2 + \beta ue + w_2 - \beta \hat{\Gamma}_1^u(w)\right). \end{aligned} \quad (4.61)$$

It is well known that the maps ϕ, ψ are both Lipschitz continuous maps in uniform topology (see [25] for example). More precisely, for $x, x' \in \mathcal{D}([0, \infty), \mathbb{R})$ and $\|x\|_T \doteq \sup_{0 \leq s \leq T} |x(s)|$, we have

$$\|\phi(x) - \phi(x')\|_T \leq C_0 \|x - x'\|_T, \quad \|\psi(x) - \psi(x')\|_T \leq C_0 \|x - x'\|_T, \quad (4.62)$$

for some $C_0 > 0$. Using the representations in (4.61) above, one can verify that the Skorohod maps defined in Lemma 4.1 are continuous functions in the metric of uniform convergence on compacts in the following sense: For all $T > 0$

$$\begin{aligned} &|u_i^n - u_i| \rightarrow 0, \quad \|w_i^n - w_i\|_T \rightarrow 0 \text{ as } n \rightarrow \infty, \quad i = 1, 2 \\ \text{implies} \quad &\|\Gamma_i^{u^n}(w_1^n, w_2^n) - \Gamma_i^u(w_1, w_2)\|_T \rightarrow 0, \quad \|\hat{\Gamma}_i^{u^n}(w_1^n, w_2^n) - \hat{\Gamma}_i^u(w_1, w_2)\|_T \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for } i = 1, 2. \end{aligned} \quad (4.63)$$

The above continuity property will be crucial for establishing some of the convergence results in the proofs below.

4.2 Weak convergence analysis and proof of Theorems 2.7 and 2.8

Note that for any admissible policy $\{\mu^n\}$, there exists $u \geq 0$, such that \tilde{u}^n, u^n both converge to u , as n tends to infinity. To simplify notation, we will use the following abbreviation for this section: $\Gamma_i^n \doteq \Gamma_i^{(\tilde{u}^n, u^n)}$, $\hat{\Gamma}_i^n \doteq \hat{\Gamma}_i^{(\tilde{u}^n, u^n)}$, $\Gamma_i \doteq \Gamma_i^{(u, u)}$, $\hat{\Gamma}_i \doteq \hat{\Gamma}_i^{(u, u)}$, $i = 1, 2$. We start with the following lemma, which describes equivalent representations of the cost functionals in the queueing network control problems as well as the Brownian control problems above.

Lemma 4.2 *The long-run average cost functionals for the queueing network and the BCP in (2.9) and (3.18) respectively have the following representation:*

$$\begin{aligned} \hat{I}(x_1, x_2, \{\mu^n\}) &= \liminf_{n \rightarrow \infty} \left[h(u^n) + \hat{\gamma}(\{\mu^n\}) + \limsup_{T \rightarrow \infty} \frac{1}{T} E \hat{L}_2^n(T) \right], \\ \text{where } \hat{\gamma}(\{\mu^n\}) &= \left(1 - \frac{\lambda_N^n}{\mu^n} \right) \sum_{i=0}^{\infty} c \left(\frac{i}{\sqrt{n}} \right) \left(\frac{\lambda_N^n}{\mu^n} \right)^i, \end{aligned} \quad (4.64)$$

$$I(x_1, x_2, u) = h(u) + \gamma(u), \quad \text{where } \gamma(u) = \frac{2u}{\sigma_1^2} \int_0^{\infty} c(x) e^{-\frac{2ux}{\sigma_1^2}} dx. \quad (4.65)$$

The infinite horizon discounted cost functionals for the queueing network and the BCP in (2.10) and (3.32) respectively have the following representation:

$$\hat{J}(x_1, x_2, \rho, \{\mu^n\}) = \liminf_{n \rightarrow \infty} \frac{h(u^n)}{\alpha} + E \int_0^{\infty} \alpha e^{-\alpha t} \left[\int_0^t (\hat{X}_1^n(s))^2 ds + k \hat{L}_2^n(t) \right] dt, \quad (4.66)$$

$$J(x_1, x_2, u) = \frac{h(u)}{\alpha} + E \int_0^{\infty} \alpha e^{-\alpha t} \left[\int_0^t (X_1(s))^2 ds + k L_2(t) \right] dt, \quad (4.67)$$

Proof. To verify (4.64), first note that for each fixed $n \geq 1$, \hat{X}_1^n is a jump Markov process with state-space $\mathcal{L}^n \doteq \left\{ \frac{j}{\sqrt{n}} : j = 0, 1, 2, \dots \right\}$, and jump rates given by

$$\begin{aligned} \mathcal{Q}^n(i, j) &= n\lambda_N^n, \text{ if } j = i + \frac{1}{\sqrt{n}}, i \in \mathcal{L}^n \\ &= n\mu^n, \text{ if } j = i - \frac{1}{\sqrt{n}}, i \in \mathcal{L}^n \setminus \{0\} \\ &= 0, \text{ otherwise.} \end{aligned}$$

Straightforward calculations (solving the balance equation) yields that the invariant distribution for \hat{X}_1^n is that of a random variable X_∞^n , such that $\sqrt{n}X_\infty^n$ follows a *Geometric* distribution with parameter $a_n = 1 - (\lambda_N^n/\mu^n)$. Therefore it follows that it follows that

$$\lim_{T \rightarrow \infty} \left[\frac{1}{T} E \int_0^T c(\hat{X}_1^n(s)) ds \right] = E[X_\infty^n].$$

This proves that the representation in (4.64) is true. The representation in (4.65) follows from Proposition 3.1 and Lemma 3.2. The proof of the representations of the discounted cost functionals in (4.66)-(4.67) are standard and similar to that of Lemma 4.4 of [15]. This completes the proof of the lemma. \blacksquare

Proposition 4.3 *The processes \hat{X}_1^n, \hat{X}_2^n in (4.57)-(4.58) satisfies*

$$(\hat{X}_1^n, \hat{X}_2^n, \hat{L}_1^n, \hat{L}_2^n) = \left(\Gamma_1^n(W^n), \Gamma_2^n(W^n), \hat{\Gamma}_1^n(W^n), \hat{\Gamma}_2^n(W^n) \right). \quad (4.68)$$

For the processes \hat{W}_1^n and \hat{W}_2^n defined in (4.56), the following convergence holds:

$$W^n \equiv (\hat{W}_1^n, \hat{W}_2^n) \Rightarrow (W_1, W_2) \equiv W, \quad (4.69)$$

where W is two-dimensional Brownian motion as described in (3.17). Also, the following holds.

$$\begin{aligned} (\hat{X}_1^n, \hat{X}_2^n, \hat{L}_1^n, \hat{L}_2^n) &\Rightarrow (X_1, X_2, L_1, L_2) \text{ and } n \rightarrow \infty, \\ \text{where } (X_1, X_2, L_1, L_2) &\doteq \left(\Gamma_1(W), \Gamma_2(W), \hat{\Gamma}_1(W), \hat{\Gamma}_2(W) \right), \end{aligned} \quad (4.70)$$

and $(W_1, W_2, X_1, X_2, L_1, L_2)$ satisfies all the conditions on the processes involved in defining the BCPs in (3.17).

Proof. From the representations of \hat{X}_1^n, \hat{X}_2^n in (4.57)-(4.58) we get (4.68), using the properties of the Skorohod map defined in Lemma 4.1.

To verify (4.69), first note that from the functional central limit theorem for Poisson processes and Assumption 2.1, we have

$$(\hat{M}_1^n, \hat{M}_2^n, \hat{M}_4^n) \Rightarrow (M_1, M_2, M_4) \text{ as } n \rightarrow \infty, \quad (4.71)$$

where M_1, M_2, M_4 are three independent Brownian motion starting from 0, with drift 0 and diffusion parameters $\bar{\lambda}_N, \bar{\mu}, \bar{\lambda}_R$ respectively. Also note that, if we define

$$\zeta^n(t) \doteq \frac{\Phi^n([nt]) - n\beta[nt]}{\sqrt{n}}, \text{ for } t \geq 0, n \geq 1,$$

then it follows that

$$\zeta^n(\cdot) \Rightarrow Z(\cdot), \text{ as } n \rightarrow \infty, \quad (4.72)$$

where Z is a driftless Brownian motion, starting from zero with diffusion parameter $\beta(1 - \beta)$. From (4.71) it follows that

$$\left(\frac{1}{\sqrt{n}} M_2^n, \frac{1}{\sqrt{n}} M_2^n(\cdot) \right) \Rightarrow (0, 0), \text{ as } n \rightarrow \infty. \quad (4.73)$$

Also note that from the (4.68) and properties of Skorohod maps in (4.61), (4.60), Assumption 2.4, (4.56) and (4.73) that

$$\frac{1}{\sqrt{n}} \hat{L}_1^n(\cdot) = \psi \left(\frac{x_1^n}{\sqrt{n}} - (\mu^n - \lambda_N^n) e(\cdot) + \left[\frac{1}{\sqrt{n}} M_1^n(\cdot) - \frac{1}{\sqrt{n}} M_2^n(\cdot) \right] \right) \Rightarrow 0, \text{ as } n \rightarrow \infty. \quad (4.74)$$

Hence, using Assumption 2.1, we have

$$\theta^n(t) = \frac{1}{n} N_2^n \left(\int_0^t \mu^n I_{\{\hat{X}_1^n(s) > 0\}} ds \right) = \frac{1}{\sqrt{n}} \hat{M}_2^n(t) + \mu^n t - \frac{1}{\sqrt{n}} \hat{L}_1^n(t) \Rightarrow \bar{\mu} e(\cdot), \text{ as } n \rightarrow \infty, \quad (4.75)$$

using (4.73) and (4.74). Hence by a random time change theorem (see Sec. 14 of [6]) we obtain

$$\hat{M}_3^n(\cdot) = \zeta^n(\theta^n(\cdot)) \Rightarrow Z(\bar{\mu} \cdot) \doteq M_3(\cdot), \text{ as } n \rightarrow \infty, \quad (4.76)$$

where M_3 is a driftless Brownian motion starting from 0 with diffusion parameter $\bar{\mu}\beta(1 - \beta)$. It is easy to verify that M_3 is independent of M_1 and M_4 and since $\langle \hat{M}_2^n, \hat{M}_3^n \rangle = -\beta \langle \hat{M}_2^n, \hat{M}_2^n \rangle$, we have from (4.71) that $\langle M_2, M_3 \rangle = -\beta \bar{\mu}$. Hence, defining $W_1 = M_1 - M_2, W_2 \doteq M_3 - M_4$, we see that $W = (W_1, W_2)$ satisfies the description in (3.17). Also, by the convergence results in (4.71) and (4.76) as well as the fact that the limits are continuous, we get that

$$\hat{W}_1^n, \hat{W}_2^n \Rightarrow (W_1, W_2), \text{ as } n \rightarrow \infty.$$

This completes the proof of (4.69).

From (4.69) and using Skorohod embedding theorem, we have that $\hat{W}^n \rightarrow W$ uniformly on compacts, and hence by (4.68) and (4.63), the claim in (4.70) follows. The last statement of the proposition follows trivially from the properties of the Skorohod map defined in Lemma 4.1. \blacksquare

The following basic lemma will be used in our proof of the main result. A proof can be found in the Appendix section at the end of the paper.

Lemma 4.4 *Let $\{a_n\}$ be a sequence such that $a_n \rightarrow a$ as $n \rightarrow \infty$ and $c(\cdot)$ be the cost function used in our analysis. Then the following holds*

$$a_n \sum_{k=0}^{\infty} c\left(\frac{k}{\sqrt{n}}\right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \rightarrow a \int_0^{\infty} c(x) e^{-ax} dx, \text{ as } n \rightarrow \infty. \quad (4.77)$$

Proposition 4.5 [Estimates] *The following estimates hold: There exists constants $C_1, C_2 > 0$ such that for all $n \geq 1$ and $t \geq 0$*

$$E \left[\sup_{0 \leq s \leq t} |\hat{X}_1^n(s)|^4 \right] \leq C_1 (1 + t^2 + t^4) \quad (4.78)$$

$$E \left[\left(\hat{L}_2^n(t) \right)^2 \right] \leq C_2 (1 + t + |u^n - \tilde{u}^n|^2 t^2). \quad (4.79)$$

Proof. From the representation of \hat{X}_1^n in (4.70) and (4.61)-(4.62), we have

$$\begin{aligned} E \left[\sup_{0 \leq s \leq t} |\hat{X}_1^n(s)|^4 \right] &\leq C_0^4 E \left[\sup_{0 \leq s \leq t} |x_1^n - \tilde{u}^n s + \hat{W}_1^n(s)|^4 \right] \\ &\leq CE \left[(x_1^n)^4 + (\tilde{u}^n)^4 t^4 + E \left(\sup_{0 \leq s \leq t} |\hat{W}_1^n(s)|^4 \right) \right], \end{aligned} \quad (4.80)$$

for some constant $C > 0$, independent of n and t . Since $\{x_1^n\}$ and $\{\tilde{u}^n\}$ are both convergent sequences, by Doob's inequality for the martingale $\hat{W}_1^n(\cdot)$, and using the fact that $E(\hat{W}_1^n(t))^4 \leq C't^2$ (for some $C' > 0$) we have the proof of the first estimate of the proposition.

For the second estimate, note that from (4.57), we have

$$\begin{aligned} E \left[\sup_{0 \leq s \leq t} |u^n s - \hat{L}_1^n(s)|^2 \right] &\leq C \left[|u^n - \tilde{u}^n|^2 t^2 + \sup_{0 \leq s \leq t} |\tilde{u}^n s - \hat{L}_1^n(s)|^2 \right] \\ &\leq C \left[|u^n - \tilde{u}^n|^2 t^2 + (x_1^n)^2 + E \left(\sup_{0 \leq s \leq t} |\hat{W}_1^n(s)|^2 \right) + E \left(\sup_{0 \leq s \leq t} |\hat{X}_1^n(s)|^2 \right) \right] \\ &\leq C (|u^n - \tilde{u}^n|^2 t^2 + 1 + t + t^2), \end{aligned} \quad (4.81)$$

where the last estimate follows using arguments similar to those used in obtaining (4.80) (and $C > 0$ represents different generic constant independent of n and t . The value of this constant varies from line to line). Now, from the representation of \hat{L}_1^n in (4.57), we have

$$E \left[\left(\hat{L}_2^n(t) \right)^2 \right] \leq C_0^2 \left[(x_2^n)^2 + E \left(\sup_{0 \leq s \leq t} |\hat{W}_2^n(s)|^2 \right) + \beta E \left(\sup_{0 \leq s \leq t} |u^n s - \hat{L}_1^n(s)|^2 \right) \right].$$

The second estimate now follows from (4.81), the Doob's inequality for the martingale $\hat{W}_2^n(\cdot)$, and the fact that $E(\hat{W}_2^n(t))^2 \leq C't$, for some $C' > 0$. \blacksquare

Now we prove the main theorems of the paper, viz. Theorems 2.7 and 2.8 using the results above.

Proof of Theorem 2.7: First we prove the asymptotic analysis of our proposed policy for the ergodic cost problem. Since for any admissible policy $\{\mu^n\}$, the corresponding $\{u^n\}$ converges to some $u \geq 0$, we have from the continuity of $h(\cdot)$ that

$$h(u^n) \rightarrow h(u), \text{ as } n \rightarrow \infty. \quad (4.82)$$

Also, note that for such policies, $a_n \doteq \sqrt{n}(1 - \frac{\lambda_N^n}{\mu^n}) = (\tilde{u}^n/\mu^n)$ converges to $a \doteq \frac{u}{\mu} = \frac{2u}{\sigma_1^2}$, since $\sigma_1^2 = \bar{\lambda}_N + \bar{\mu} = 2\bar{\mu}$ by Assumption 2.4. Therefore, by Lemma 4.4, we have

$$\hat{\gamma}^n(\{\mu^n\}) \rightarrow \gamma(u), \text{ as } n \rightarrow \infty. \quad (4.83)$$

By Proposition 4.5, we have that

$$\liminf_{n \rightarrow \infty} \left[\limsup_{T \rightarrow \infty} \frac{1}{T} E \hat{L}_2^n(T) \right] \leq \liminf_{n \rightarrow \infty} \left[\limsup_{T \rightarrow \infty} \frac{1}{T} \sqrt{C_2(1 + T + |u^n - \tilde{u}^n|^2 T^2)} \right] = 0. \quad (4.84)$$

Hence by Lemma 4.2, we have that for any admissible control policy $\{\mu^n\}$, with u^n converging to some $u \geq 0$,

$$\hat{I}(x_1, x_2, \{\mu^n\}) = I(x_1, x_2, u) \quad (4.85)$$

Note that from the construction of our proposed policy $\{\mu_1^{n,*}\}$, we have that the corresponding $\{u^{n,*}\}$ converges to $u_1^* \geq 0$, where u_1^* is as in Theorem 3.3. Hence we have that from Theorem 3.3 and (4.85) that for any admissible policy $\{\mu^n\}$

$$\hat{I}(x_1, x_2, \{\mu^n\}) = I(x_1, x_2, u) \geq I(x_1, x_2, u_1^*) = \hat{I}(x_1, x_2, \{\mu_1^{n,*}\}). \quad (4.86)$$

This proves the asymptotic optimality of our proposed policy for the queueing network problem with long run average (ergodic) cost. \blacksquare

Proof of Theorem 2.8: Now we prove the optimality result with for the infinite horizon discounted cost problem. Note that for any admissible control policy $\{\mu^n\}$ with the corresponding $\{u^n\}$ converging to some $u \geq 0$, we have from Proposition 4.3 that

$$\int_0^\cdot (\hat{X}_1^n(s))^2 ds \Rightarrow \int_0^\cdot (X_1(s))^2 ds, \text{ as } n \rightarrow \infty. \quad (4.87)$$

This follows from the fact that $\hat{X}_1^n \Rightarrow X_1$ and the function $\int_0^\cdot x(s)^2 ds$ is a continuous map on $\mathcal{D}([0, \infty), \mathbb{R})$ with respect to the uniform metric on compacts, and hence continuous mapping theorem applies. Combining the last part of Proposition 4.3 and (4.87), we have for all fixed $t \geq 0$

$$\left[\int_0^t (\hat{X}_1^n(s))^2 ds + k \hat{L}_2^n(t) \right] \Rightarrow \left[\int_0^t (X_1(s))^2 ds + k L_2(t) \right], \text{ as } n \rightarrow \infty. \quad (4.88)$$

Observe that from Proposition 4.5, we have for each fixed $t \geq 0$,

$$E \left[\int_0^t (\hat{X}_1^n(s))^2 ds + k \hat{L}_2^n(t) \right]^2 \leq C_3[1 + t^2 + t^4], \text{ for all } n \geq 1, \quad (4.89)$$

where $C_3 > 0$ is a constant independent of n and t . From (4.88) and (4.89) we get that for each fixed $t \geq 0$,

$$E \left[\int_0^t (\hat{X}_1^n(s))^2 ds + k \hat{L}_2^n(t) \right] \rightarrow E \left[\int_0^t (X_1(s))^2 ds + k L_2(t) \right], \text{ as } n \rightarrow \infty. \quad (4.90)$$

Now from (4.89), it is easy to verify that

$$\int_0^\infty \alpha e^{-\alpha t} E \left[\int_0^t (\hat{X}_1^n(s))^2 ds + k \hat{L}_2^n(t) \right]^2 dt \leq C_4 \text{ for all } n \geq 1, \quad (4.91)$$

where $C_4 > 0$ is a constant independent of n and t . This bound together with the convergence in (4.90) implies that

$$\int_0^\infty \alpha e^{-\alpha t} E \left[\int_0^t (\hat{X}_1^n(s))^2 ds + k \hat{L}_2^n(t) \right] dt \rightarrow \int_0^\infty \alpha e^{-\alpha t} E \left[\int_0^t (X_1(s))^2 ds + k L_2(t) \right] dt, \text{ as } n \rightarrow \infty. \quad (4.92)$$

Hence by Lemma 4.2, we have that for any admissible control policy $\{\mu^n\}$, with u^n converging to some $u \geq 0$,

$$\hat{J}(x_1, x_2, \{\mu^n\}) = J(x_1, x_2, u). \quad (4.93)$$

Using the same arguments as in obtaining (4.85), we get

$$\hat{J}(x_1, x_2, \{\mu^n\}) = J(x_1, x_2, u) \geq J(x_1, x_2, u_2^*) = \hat{I}(x_1, x_2, \{\mu_2^{n,*}\}), \quad (4.94)$$

where $u_2^* \geq 0$ is the optimal drift for the BCP as given in Theorem 3.12. This completes the proof. \blacksquare

5 Numerical Analysis

To numerically compute the optimal controls u^* of Theorems 3.3 and 3.12, here we implement a numerical calculations with specific choices of parameters. Consider,

$\alpha = 1, k = 1, h(u) = u^2, c(x) = x^2, \bar{\lambda}_N = 10, \bar{\lambda}_R = 1, \beta = 0.1, \bar{\mu} = 10, x_1 = 1200, x_2 = 1.$

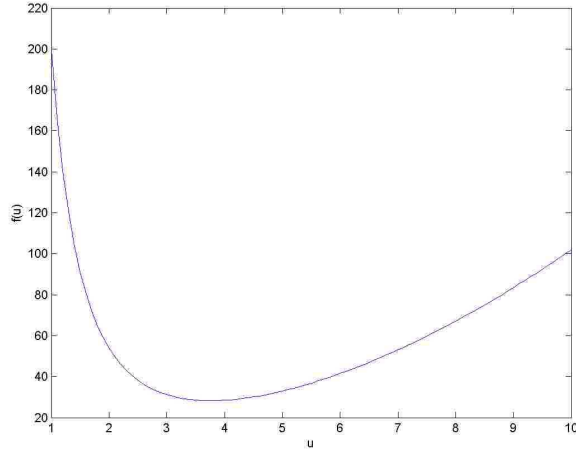


Figure 2: Cost $I(\cdot)$ as a function of u , minimized at $u^* = 3.8$ and $I(u^*) = 28.3$.

For the long-run average (ergodic) cost problem, recall the function $F(\cdot)$ defined in (3.31) (note that $I(x_1, x_2, u) = F(u)$ from (3.30) for any choice of (x_1, x_2)).

$$F(u) = \int_0^\infty e^{-y} \left[u^2 + \frac{\sigma_1^4}{4u^2} y^2 \right] dy = u^2 + \frac{\sigma_1^4}{2u^2}.$$

By numerical methods (as suggested in the last part of subsection 3.1), we obtain $u^* = 3.8, I(u^*) = 28.3$. The graph above show how $I(\cdot)$ behaves as a function of u :

For the discounted cost control problem, we use the same values for the parameters. In this case, we generate two independent standard Brownian motion processes $B_1(t)$ and $B_2(t)$ for $t = 0, 0.1, 0.2, \dots, 100$. Let $W_1(t) = B_1(t)$ and $W_2(t) = -rB_1(t) + \sqrt{1 - r^2}B_2(t)$, where $r = \frac{\bar{\mu}\beta}{\sqrt{(\bar{\lambda}_N + \bar{\mu})(\bar{\lambda}_R + \bar{\mu}\beta(1-\beta))}}$. Computing X_1, L_1, L_2 from (W_1, W_2) using (4.70) of Proposition 4.3, we calculate calculate $J(u)$ for $u = 0, 0.1, 0.2, \dots, 2$, and obtain the optimal u^* . In this case, optimal we get $u^* = 1.2$ and $J(u^*) = 21.2$. The following graph describes the behavior of the cost function J as a function of u .

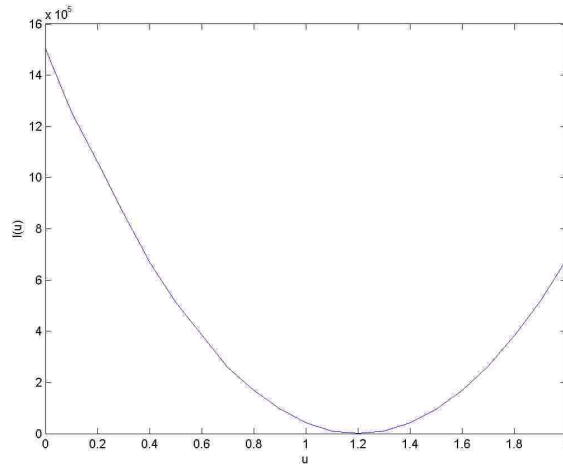


Figure 3: Cost $J(\cdot)$ as a function of u , minimized at $u^* = 1.2$ and $J(u^*) = 21.2$.

6 Discussion:

7 Appendix

Proof of Lemma 4.4 Consider a sequence of random variables $\{Y_n\}$, where for each $n \geq 1$, X_n follows a *Geometric* distribution with parameter $\frac{a_n}{\sqrt{n}}$, and let X be an *Exponential* random variable with parameter a . Since $a_n \rightarrow a$ as $n \rightarrow \infty$, it is easy to verify that

$$\left(1 - \frac{a_n}{\sqrt{n}}\right)^{\sqrt{n}} \rightarrow e^{-a}, \text{ as } n \rightarrow \infty.$$

Straightforward calculations using the above yields $\frac{X_n}{\sqrt{n}} \Rightarrow X$. Therefore for all $M > 0$, using the continuity of $c(\cdot)$, we have

$$E \left[c \left(\frac{X_n}{\sqrt{n}} \right) I_{\left\{ \frac{X_n}{\sqrt{n}} \leq M \right\}} \right] \rightarrow E [c(X) I_{\{X \leq M\}}] \text{ as } n \rightarrow \infty.$$

Hence, for all $M > 0$,

$$a_n \sum_{k=0}^{\sqrt{n}M} c \left(\frac{k}{\sqrt{n}} \right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \rightarrow a \int_0^M c(x) e^{-ax} dx, \text{ as } n \rightarrow \infty. \quad (7.95)$$

Note that there exists $M_0 > 0$ such that $c(x)e^{-ax}$ decreases as a function of x for all $x \geq M_0$. Fix any $\epsilon > 0$. There exists $M \geq M_0$ such that

$$a \int_M^{\infty} c(x) e^{-ax} dx < \frac{\epsilon}{3}. \quad (7.96)$$

Since $M \geq M_0$, we have that the integrand below is decreasing and so from (7.96)

$$\sum_{k=\sqrt{n}M+1}^{\infty} c \left(\frac{k}{\sqrt{n}} \right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \leq \int_M^{\infty} c(x) e^{-ax} dx \doteq b_M. \quad (7.97)$$

Fix $n_0(M) \geq 1$ such that $|a_n - a| \leq \epsilon/(3b_M)$, for all $n \geq n_0(M)$. From (7.96) and (7.97) we have for all $n \geq n_0(M)$,

$$\begin{aligned} a_n \sum_{k=\sqrt{n}M+1}^{\infty} c \left(\frac{k}{\sqrt{n}} \right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} &\leq |a_n - a| \sum_{k=\sqrt{n}M+1}^{\infty} c \left(\frac{k}{\sqrt{n}} \right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \\ &\quad + a \sum_{k=\sqrt{n}M+1}^{\infty} c \left(\frac{k}{\sqrt{n}} \right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} \\ &\leq |a_n - a| b_M + a \int_M^{\infty} c(x) e^{-ax} dx \leq \frac{2\epsilon}{3}. \end{aligned} \quad (7.98)$$

Hence, from (7.96) and (7.98) we have that for all $\epsilon > 0$, there exists large $M > 0$, such that

$$\left| a_n \sum_{k=\sqrt{n}M+1}^{\infty} c \left(\frac{k}{\sqrt{n}} \right) \left(1 - \frac{a_n}{\sqrt{n}}\right)^k \frac{1}{\sqrt{n}} - a \int_M^{\infty} c(x) e^{-ax} dx \right| < \epsilon, \quad (7.99)$$

for $n \geq n_0(M)$. From (7.95) and (7.99), the proof of the lemma is complete. \blacksquare

Acknowledgments: Arka Ghosh is supported by the NSF grant DMS-0608669. Ananda Weerasinghe is supported by the ARO grant W911NF0510032. We are grateful for their support.

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